

# Eigen Wavefunctions of a Charged Particle Moving in a Self-Linking Magnetic Field

Dah-Wei Chiou\* and Dung-Hai Lee†

*Department of Physics*

*University of California at Berkeley*

*Berkeley, CA 94720 USA*

Wu-Yi Hsiang‡

*Department of Mathematics*

*University of California at Berkeley*

*Berkeley, CA 94720 USA*

## Abstract

In this paper we solve the one-particle Schrödinger equation in a magnetic field whose flux lines exhibit mutual linking. To make this problem analytically tractable, we consider a high-symmetry situation where the particle moves in a three-sphere ( $S^3$ ). The vector potential is obtained from the Berry connection of the two by two Hamiltonian  $H(\mathbf{r}) = \hat{h}(\mathbf{r}) \cdot \vec{\sigma}$ , where  $\mathbf{r} \in S^3$ ,  $\hat{h} \in S^2$  and  $\vec{\sigma}$  are the Pauli matrices. In order to produce linking flux lines, the map  $\hat{h} : S^3 \rightarrow S^2$  is made to possess nontrivial homotopy. The problem is exactly solvable for a particular mapping  $(\hat{h})$ . The resulting eigenfunctions are  $SO(4)$  spherical harmonics, the same as those when the magnetic field is absent. The highly nontrivial magnetic field lifts the degeneracy in the energy spectrum in a way reminiscent of the Zeeman effect.

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\*Electronic address: [dwchiou@socrates.berkeley.edu](mailto:dwchiou@socrates.berkeley.edu)

†Electronic address: [dunghai@socrates.berkeley.edu](mailto:dunghai@socrates.berkeley.edu)

‡Electronic address: [hsiang@math.berkeley.edu](mailto:hsiang@math.berkeley.edu)

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## I. INTRODUCTION

It is remarkable that the basis of some of the most fundamental theories of nature, such as general relativity and gauge theories, is the mathematics of fiber bundles. A fiber bundle is locally characterized by its curvature tensor (the strength of the gauge field), and globally by its topological invariants.

For example the quantization of the magnetic charge of a Dirac monopole [1, 2] is equivalent to the quantization of the first Chern number,  $\int_{S^2} ch_1(\mathcal{F})$ , for the  $U(1)$  bundle over  $S^2$  (See [3] and Subsection III A below). Another well-known example is the  $SU(2)$  instanton [4]. The instanton number is given by the integral of the second Chern character,  $\int_{S^4} ch_2(\mathcal{F})$ , for the  $SU(2)$  bundle over  $S^4$  [3].

Under the influence of such topologically nontrivial background gauge field, the dynamics of the particles that interact with it is profoundly affected. For example Wu and Yang have shown that in the presence of a Dirac monopole the orbital angular momentum quantum number of a charge  $e$  particle can become odd integer multiples of  $\hbar$  [2]. Under such condition we need to generalize “wave function” to “*wave section*” which is patch-wise defined in space [5]. The eigensections of the angular momentum are called *monopole harmonics*  $Y_{q,l,m}$ , where  $q = e \times$  the magnetic charge. The allowed values of  $q$  are multiples of half-integers, and the allowed values of  $l$  and  $m$  are:  $l = |q|, |q| + 1, |q| + 2, \dots$  and  $m = -l, -l + 1, \dots, l$ . Thus half-integer  $q$  gives half-integer  $l$ , which means  $Y_{q,l,m}$  can form a complete representation of  $SU(2)$  whereas ordinary spherical harmonics  $Y_{l,m}$  can only represent  $SO(3)$ . Therefore, without introducing the spinor structure, applying nontrivial field strength alone can lift the wave function representation to fulfill the universal covering group.

As stated above, the first Chern number counts the number of fundamental monopoles enclosed by a 2-dimensional closed surface. In a 3-dimensional space, field configurations with the same Chern invariant may have different linking topologies for the loops of the magnetic lines. *Chern-Simons invariant* further distinguishes this topology. In this paper, in the spirit of Wu and Yang, we investigate the effect of self-linking magnetic field on charged particle dynamics. To make this problem tractable analytically, we consider a high-symmetry situation where the particle moves in a three-sphere ( $S^3$ ). The vector potential is obtained from the Berry connection of the two by two Hamiltonian  $H(\mathbf{r}) = \hat{h}(\mathbf{r}) \cdot \vec{\sigma}$ , where  $\mathbf{r} \in S^3$ ,  $\hat{h} \in S^2$  and  $\vec{\sigma}$  are the Pauli matrices [7]. In order to produce linking flux lines, the map  $\hat{h} : S^3 \rightarrow S^2$  is made to possess nontrivial homotopy. For a particular mapping ( $\hat{h}$ ) the problem is exactly solvable. The resulting eigenfunctions are  $SO(4)$  spherical harmonics, the same as those when the magnetic field is absent. The highly nontrivial magnetic field lifts the degeneracy in the energy spectrum in a way reminiscent of the Zeeman effect.

This paper is organized as follows. In Section II, we briefly review the physics of Berry’s phase. In Section III, we study the topological classifications by Chern and Chern-Simons

invariants. Later, in Section IV, with the help of Berry connection, we construct the Schrödinger equation in the presence of magnetic field with nontrivial Chern-Simons invariant. We solve the Schrödinger equation without magnetic field in Section V and get  $SO(4)$  spherical harmonics. Finally, in Section VI, the Schrödinger equation with magnetic field of nonzero Chern-Simons invariant is solved. We conclude in Section VII with a discussion of our results.

## II. BERRY'S PHASE AND ITS GEOMETRIC NATURE

In quantum mechanics, the overall phase of a wave function is often regarded as an irrelevant factor. Berry pointed out that the phase may have observable consequences if the system undergoes an adiabatic change [8]. In fact, Berry's phase has a deep geometrical meaning with gauge nature [9] and can be used to give the electromagnetic potential as showed in Example 1.

### A. Geometric (Berry's) Phase

Consider a Hamiltonian  $H(\mathbf{x})$  which depends on an external set of parameters  $\mathbf{x} = (x_1, \dots, x_n)$ . If  $\mathbf{x}(t)$  changes “adiabatically” (slowly) along a closed path  $C : \mathbf{x}(t) \ t \in [t_i, t_f]$ , with  $\mathbf{x}(t_i) = \mathbf{x}(t_f)$  in the parameter space, then the solution of the equation  $i\partial_t|\psi(t)\rangle = H(\mathbf{x}(t))|\psi(t)\rangle$  is

$$|\psi(t_f)\rangle = e^{-i \int_{t_i}^{t_f} dt E(\mathbf{x}(t))} e^{i\gamma_C} |\psi(t_i)\rangle, \quad (1)$$

where

$$\gamma_C = \oint_C A_\mu(x) dx^\mu, \quad \text{with} \quad A_\mu(x) = -i\langle\psi(x)|\partial_\mu\psi(x)\rangle. \quad (2)$$

In Eq. (2),  $|\psi(\mathbf{x})\rangle$  is a *normalized, non-degenerate* eigenstate satisfying

$$H(\mathbf{x})|\psi\rangle = E(\mathbf{x})|\psi(\mathbf{x})\rangle. \quad (3)$$

In addition, the phase of  $|\psi(\mathbf{x})\rangle$  is chosen so that  $\partial_\mu|\psi(\mathbf{x})\rangle$  is defined for  $\mathbf{x} \in C$ .

Therefore, after  $x(t)$  completes a closed circuit, the eigenstate  $|\psi\rangle$  returns to its initial value with an additional phase, of which  $e^{-i \int dt E}$  is due to the time revolution while the extra phase  $e^{i\gamma_C}$  is called *geometric (Berry's) phase*.

## B. The Berry Connection and Berry Curvature; the Abelian Case

If we consider a local (gauge) transformation,  $|\psi(\mathbf{x})\rangle \rightarrow e^{i\theta(\mathbf{x})}|\psi(\mathbf{x})\rangle$ , and use  $e^{i\theta(\mathbf{x})}|\psi(\mathbf{x})\rangle$  to compute  $A_\mu$ , we get

$$A_\mu(\mathbf{x}) \rightarrow A_\mu(\mathbf{x}) + \partial_\mu \theta(\mathbf{x}), \quad (4)$$

which shows that  $A_\mu(x)$  transforms like a gauge field. In the literature it is often called *Berry connection*. By drawing analogy with electromagnetism we can define the *Berry curvature* as :

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (5)$$

The Berry curvature is gauge-invariant and its integral over the surface bounded by the closed path  $C$  gives the Berry phase:

$$\gamma_C = \int_{surface} F_{\mu\nu} dx^\mu dx^\nu. \quad (6)$$

**Example 1** Consider the following  $2 \times 2$  Hamiltonian:

$$H(x, y, z) = \vec{r} \cdot \vec{\sigma} = x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}, \quad (7)$$

where the  $\sigma$ 's are Pauli matrices.  $H$  has two eigenvalues  $\pm\sqrt{x^2 + y^2 + z^2} \equiv \pm r$  and two eigenvectors

$$|\psi_\pm(x, y, z)\rangle = \frac{1}{N_\pm} \begin{pmatrix} \frac{z \pm r}{x + iy} \\ 1 \end{pmatrix}. \quad (8)$$

Taking  $|\psi_\pm(x, y, z)\rangle$  to compute  $A_\mu^\pm = -i\langle\psi_\pm|\partial_\mu\psi_\pm\rangle$ , we obtain

$$A_x = \frac{-y}{2r(r \mp z)}, \quad A_y = \frac{x}{2r(r \mp z)}, \quad A_z = 0, \quad (9)$$

from which  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is obtained:

$$F_{xy} = \mp \frac{z}{2r^3}, \quad F_{zx} = \mp \frac{y}{2r^3}, \quad F_{yz} = \mp \frac{x}{2r^3}. \quad (10)$$

If we interpret  $x, y, z$  as the spatial space coordinates, then  $\varepsilon^{\mu\nu\lambda} F_{\nu\lambda}$  is precisely the magnetic field produced by a monopole at  $r = 0$ .

Remark: The connection  $A_\mu$  in Eq. (9) is singular in the north (south) pole direction ( $z = \pm r$ ). In order to derive a non-singular  $A_\mu$ , we can use Eq. (8) in the southern (northern)

hemisphere, and

$$|\psi_{\pm}(x, y, z)\rangle = \frac{1}{N_{\pm}} \begin{pmatrix} 1 \\ \frac{x+iy}{r\pm z} \end{pmatrix} \quad (11)$$

in the norther (southern) hemisphere. The vector potential derived from Eq. (8) and Eq. (11) differs by a gauge transformation along the equator [2]. (Also see Example 3 below.)

### C. The Berry Connection and Berry Curvature; the Non-Abelian Case

This can be extended to non-Abelian case if we consider  $n$ -fold degenerate eigenstates instead of a non-degenerate one. Let  $|\psi^{\alpha}\rangle$ ,  $\alpha = 1, \dots, n$  represent the members of the degenerate  $n$ -tuple. The Berry connection is an  $n \times n$  matrix:

$$(A_{\mu})^{\beta}_{\alpha} = -i\langle\psi_{\alpha}|\partial_{\mu}\psi^{\beta}\rangle. \quad (12)$$

Under a local (gauge) unitary transformation in the internal space (eigenspace),  $|\psi'^{\alpha}(x)\rangle = S(x)^{\alpha}_{\beta}|\psi^{\beta}(x)\rangle$ . The non-Abelian connection then transforms as:

$$\begin{aligned} (A'_{\mu})^{\beta}_{\alpha} &= -i\langle S_{\alpha}^{\gamma}\psi_{\gamma}|\partial_{\mu}(S^{\beta}_{\delta}\psi^{\delta})\rangle \\ &= S^{\beta}_{\delta}(A_{\mu})^{\delta}_{\gamma}S_{\alpha}^{\gamma*} - i(\partial_{\mu}S^{\beta}_{\delta})(A_{\mu})^{\delta}_{\gamma}S_{\alpha}^{\gamma*}. \end{aligned} \quad (13)$$

With  $S_{\alpha}^{\gamma*} = S^{\dagger\gamma}_{\alpha}$ , this gives

$$(A'_{\mu})^{\beta}_{\alpha} = S^{\beta}_{\delta}(A_{\mu})^{\delta}_{\gamma}(S^{\dagger})^{\gamma}_{\alpha} - i(\partial_{\mu}S^{\beta}_{\delta})(A_{\mu})^{\delta}_{\gamma}(S^{\dagger})^{\gamma}_{\alpha} \quad (14)$$

or

$$A'_{\mu} = S A_{\mu} S^{\dagger} - i(\partial_{\mu}S)A_{\mu}S^{\dagger}. \quad (15)$$

The non-Abelian curvature is defined as

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - i[A_{\mu}, A_{\nu}]. \quad (16)$$

Using Eq. (15), it is simple to show that  $F_{\mu\nu}$  transforms covariantly under the gauge transformation; i.e.,

$$F'_{\mu\nu} = S F_{\mu\nu} S^{\dagger}. \quad (17)$$

**Example 2** The close connection between non-Abelian gauge theories and general relativity has been noticed and elucidated by Yang [10]. In particular, the Christoffel connection can be interpreted as the non-Abelian gauge potential and the Riemann-Christoffel curvature is proportional to the corresponding field strength [11].

In this example we show that for a given (curved)  $d$ -dimensional space there exists a  $(d+1) \times (d+1)$  matrix Hamiltonian whose Berry's connection is the Christoffel connection of the subject space and the Riemann-Christoffel curvature is the minus Berry curvature. This result allows us to further tighten the connection between geometric phase and differential geometry.

Consider a  $d$ -dimensional (curved) space  $\mathcal{M}$  as being embedded in a  $d+1$ -dimensional Euclidean space  $\mathbb{R}^{d+1}$ . At each point of  $\mathcal{M}$  there is a vector

$$|n\rangle = \begin{pmatrix} n^1 \\ \vdots \\ .. \\ \vdots \\ n^{d+1} \end{pmatrix}, \quad (18)$$

which is normal to  $\mathcal{M}$ . (Here  $n^i$  are the Cartesian components of  $\mathbf{n}$ .) In the following discussion, Latin indices (refer to the embedding space) range from 1 to  $d+1$  and Greek indices (refer to the embedded space) range from 1 to  $d$ .

Given the normal vector  $\mathbf{n}$ , it is possible to construct the following  $(d+1) \times (d+1)$  matrix

$$H \equiv |n\rangle\langle n|. \quad (19)$$

In the above,  $\langle n|$ , the dual of  $|n\rangle$ , is the  $(d+1)$ -component row vector

$$\langle n| = (n^1, \dots, n^{d+1}). \quad (20)$$

Obviously

$$\langle n|n\rangle = 1. \quad (21)$$

In both Eq. (19) and Eq. (21), matrix product is assumed. Because  $|n\rangle$  and  $\langle n|$  depend on the location in  $\mathcal{M}$ , so does  $H$ . We will take such matrix as our “Hamiltonian”.

The matrix defined in Eq. (19) has one eigenvalue  $\lambda = 1$  and eigenvalues  $\lambda = 0$  with  $d$ -fold degeneracy. The  $\lambda = 1$  eigenvector is  $\mathbf{n}$  and the  $\lambda = 0$  eigenvectors are orthogonal to

$\mathbf{n}$  and hence are tangent vectors in  $\mathcal{M}$ . Since the  $\lambda = 0$  eigenvalue is  $d$ -fold degenerate, the associated Berry connection is a  $d \times d$  matrix.

Let  $|\psi_1\rangle, \dots, |\psi_d\rangle$  be the  $d$  orthonormal eigenvectors of  $H$  satisfying

$$H|\psi_\mu\rangle = 0, \quad \mu = 1, \dots, d \quad (22)$$

Obviously,  $|\psi_\mu\rangle$  defines a local orthonormal basis in the tangent space of  $\mathcal{M}$ . As the result, an infinitesimal displacement vector is given by

$$d\mathbf{x} = dx^\mu |\psi_\mu\rangle, \quad (23)$$

which implies

$$ds^2 = g_\nu^\mu dx_\mu dx^\nu, \quad (24)$$

where

$$g_\nu^\mu = \langle \psi^\mu | \psi_\nu \rangle. \quad (25)$$

The non-Abelian Berry's connection is defined as

$$(A_\mu)^\alpha_\beta = \langle \psi^\alpha | \partial_\mu | \psi_\beta \rangle. \quad (26)$$

It can be shown straightforwardly that

$$(A_\mu)^\alpha_\beta = \Gamma^\alpha_{\beta\mu}, \quad (27)$$

where  $\Gamma^\alpha_{\beta\mu}$  is the Christoffel connection. The Riemann-Christoffel curvature tensor is given by

$$R^\beta_{\nu\rho\sigma} \equiv \Gamma^\beta_{\nu\sigma,\rho} - \Gamma^\beta_{\nu\rho,\sigma} + \Gamma^\beta_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} - \Gamma^\beta_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}. \quad (28)$$

Substituting Eq. (27) into Eq. (28), we obtain

$$\begin{aligned} R^\beta_{\nu\rho\sigma} &= \{\partial_\rho A_\sigma - \partial_\sigma A_\rho + [A_\rho, A_\sigma]\}^\beta_\nu \\ &= -(F_{\rho\sigma})^\beta_\nu. \end{aligned} \quad (29)$$

### III. CHERN NUMBER AND CHERN-SIMONS INVARIANT

Given a fiber  $F$ , a structure group  $G$  and a (closed) base space  $\mathcal{M}$ , we may construct fiber bundles. A fiber bundle is locally the direct product of the fiber and the base space. However globally speaking, after non-trivial twisting of the fibers as the base space is traversed, a topologically non-trivial bundle can be produced. The question is by allowing all possible twistings how many distinct topological types of fiber bundles one can construct. In mathematics, the characteristic classes are invariants measuring the topological “non-triviality” of a fiber bundle. Among them, the *Chern classes* and the *Chern-Simons classes* are of particular interests.

#### A. Chern Number

If the bases space is a closed 2-dimensional base space  $\mathcal{M}$ , the following integral, the *Chern number* [12],

$$C = \int_{\mathcal{M}} ch_1(\mathcal{F}) = \frac{1}{8\pi} \int_{\mathcal{M}} d^2x \varepsilon^{\mu\nu} Tr[F_{\mu\nu}], \quad (30)$$

is a topological invariant. If  $C$  is nonzero the fiber bundle must have nontrivial topology. For the  $U(1)$  bundle,  $C$  is quantized as  $C = n/2$ , where  $n$  is an integer.

**Example 3** *Following example 1, consider  $H = x\sigma_x + y\sigma_y + z\sigma_z$ . However this time we restrict  $(x, y, z)$  to lie on the surface of the unit sphere in three dimension ( $S^2$ ). For each point on  $S^2$  we use the collection of all normalized eigenvectors of  $H$  with eigenvalue +1 as the fiber to construct a  $U(1)$  fiber bundle over  $S^2$ . [The structure group is  $U(1)$  because the normalized eigenvectors are related to one another by  $U(1)$  transformations.] The connection of such  $U(1)$  bundle is exactly the Berry connection discussed earlier. If we choose the polar angle  $(\theta, \phi)$  to parameterize  $S^2$ , then the eigenvector correspond to +1 eigenvalue is given by*

$$|\psi_+^S\rangle = \begin{pmatrix} e^{-i\phi} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}, \quad |\psi_+^N\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix}, \quad (31)$$

where  $|\psi_+^S\rangle$  is well-defined in the south hemisphere while  $|\psi_+^N\rangle$  is in the north hemisphere and these two differ by a gauge transformation.

The Berry connection is given by

$$\begin{aligned} A_\theta^{S/N} &= -i\langle\psi_+|\partial_\theta\psi_+\rangle = 0, \\ A_\phi^S &= -i\langle\psi_+|\partial_\phi\psi_+\rangle = -\cos^2(\theta/2), \quad A_\phi^N = \sin^2(\theta/2). \end{aligned} \quad (32)$$

As a result,  $F_{\theta\phi} = \frac{1}{2}\sin\theta$ , and

$$\begin{aligned} C &= \frac{1}{8\pi} \int d^2x \varepsilon^{\mu\nu} F_{\mu\nu}, \\ &= \frac{1}{8\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin\theta = \frac{1}{2}, \end{aligned} \quad (33)$$

since the Chern number is non-zero the  $U(1)$  fiber bundle we constructed is non-trivial.

As we discussed earlier, the Chern number of  $U(1)$  bundles must equal to  $n/2$ . In the above example we encountered a case where  $n = 1$ . Can we construct a  $2 \times 2$  Hamiltonian over  $S^2$  such that the Chern number is greater than  $1/2$ ? The answer is affirmative due to the following theorem proven by Hsiang and Lee [7].

Consider the following  $2 \times 2$  Hamiltonian defined over base space  $S^2$ :

$$H = \hat{h}(\theta, \phi) \cdot \vec{\sigma}, \quad (34)$$

where  $\vec{\sigma}$  are the Pauli matrices and  $\hat{h}(\theta, \phi)$  is a three-component unit vector field (which maps  $S^2 \rightarrow S^2$ ). This Hamiltonian has eigenvalues  $\pm 1$  and the corresponding eigenstates  $|\psi_\pm(\theta, \phi)\rangle$ . Now consider the Berry connection

$$\begin{aligned} A_\theta &= -i\langle\psi_+(\theta, \phi)|\partial_\theta|\psi_+(\theta, \phi)\rangle, \\ A_\phi &= -i\langle\psi_+(\theta, \phi)|\partial_\phi|\psi_+(\theta, \phi)\rangle, \end{aligned} \quad (35)$$

associated with, say, the + eigenstate.

**Theorem 4** *The Berry curvature,  $F_{\theta\phi}$ , associated with the above connection satisfies*

$$F_{\mu\nu} = \frac{1}{2}\hat{h} \cdot (\partial_\nu\hat{h} \times \partial_\mu\hat{h}). \quad (36)$$

**Proof.**

$$\begin{aligned} F_{\mu\nu} &= -i[\partial_\nu\langle\psi_+|\partial_\mu\psi_+\rangle - (\mu \leftrightarrow \nu)] \\ &= -i[\langle\partial_\nu\psi_+|\partial_\mu\psi_+\rangle - (\mu \leftrightarrow \nu)] \\ &= -i \sum_{n=\pm} [\langle\partial_\nu\psi_+|\psi_n\rangle\langle\psi_n|\partial_\mu\psi_+\rangle - (\mu \leftrightarrow \nu)]. \end{aligned} \quad (37)$$

Since  $\langle \psi_+ | \psi_+ \rangle = 1$ ,  $\langle \partial_\nu \psi_+ | \psi_+ \rangle + \langle \psi_+ | \partial_\nu \psi_+ \rangle = 0$  and  $\langle \partial_\nu \psi_+ | \psi_+ \rangle$  is purely imaginary. Then  $\langle \partial_\nu \psi_+ | \psi_+ \rangle \langle \psi_+ | \partial_\mu \psi_+ \rangle = \langle \partial_\mu \psi_+ | \psi_+ \rangle \langle \psi_+ | \partial_\nu \psi_+ \rangle$  is real and thus

$$F_{\mu\nu} = -i[\langle \partial_\nu \psi_+ | \psi_- \rangle \langle \psi_- | \partial_\mu \psi_+ \rangle - (\mu \leftrightarrow \nu)]. \quad (38)$$

From the first order perturbation theory,

$$\langle \psi_- | \partial_\mu \psi_+ \rangle = \frac{\langle \psi_- | \partial_\mu \hat{h} \cdot \vec{\sigma} | \psi_+ \rangle}{E_+ - E_-} = \frac{\langle \psi_- | \partial_\mu \hat{h} \cdot \vec{\sigma} | \psi_+ \rangle}{2}. \quad (39)$$

Therefore,

$$\begin{aligned} F_{\mu\nu} &= -\frac{i}{4}[\langle \psi_+ | \partial_\nu \hat{h} \cdot \vec{\sigma} | \psi_- \rangle \langle \psi_- | \partial_\mu \hat{h} \cdot \vec{\sigma} | \psi_+ \rangle - (\mu \leftrightarrow \nu)] \\ &= -\frac{i}{4} \sum_{n=\pm} [\langle \psi_+ | \partial_\nu \hat{h} \cdot \vec{\sigma} | \psi_n \rangle \langle \psi_n | \partial_\mu \hat{h} \cdot \vec{\sigma} | \psi_+ \rangle - (\mu \leftrightarrow \nu)] \\ &= -\frac{i}{4} \langle \psi_+ | [\partial_\nu \hat{h} \cdot \vec{\sigma}, \partial_\mu \hat{h} \cdot \vec{\sigma}] | \psi_+ \rangle \\ &= -\frac{i}{4} \partial_\nu h_\alpha \partial_\mu h_\beta \langle \psi_+ | [\sigma_\alpha, \sigma_\beta] | \psi_+ \rangle \\ &= \frac{1}{2} \varepsilon_{\alpha\beta\gamma} \partial_\nu h_\alpha \partial_\mu h_\beta \langle \psi_+ | \sigma_\gamma | \psi_+ \rangle, \quad \alpha, \beta, \gamma \in x, y, z. \end{aligned} \quad (40)$$

Consider  $|\psi_+\rangle = h_\gamma \sigma_\gamma |\psi_+\rangle$  and  $\sigma_\gamma^2 = 1$ ; thus, we have

$$F_{\mu\nu} = \frac{1}{2} \varepsilon_{\alpha\beta\gamma} \partial_\nu h_\alpha \partial_\mu h_\beta h_\gamma = \frac{1}{2} \hat{h} \cdot (\partial_\nu \hat{h} \times \partial_\mu \hat{h}). \quad (41)$$

■

This theorem tells us that in order to construct a Hamiltonian whose eigenvector bundle exhibits  $C > 1/2$  we need to choose  $\hat{h}$  so that

$$\mathcal{P} \equiv \frac{1}{8\pi} \int_{S^2} d^2x \varepsilon^{\mu\nu} \hat{h} \cdot (\partial_\nu \hat{h} \times \partial_\mu \hat{h}) \quad (42)$$

is greater than 1. The  $S^2 \rightarrow S^2$  identity map

$$\hat{h}(\hat{r}) = \hat{r} \quad (43)$$

gives  $\mathcal{P} = 1$  and the corresponding Hamiltonian [Eq. (34)] reduces to that discussed in Example 3. The quantity  $\mathcal{P}$ , the Pontrjagin index, is a homotopy class of the  $S^2 \rightarrow S^2$  mapping. It measures the number of times the image covers the target space. A map that gives  $\mathcal{P} > 1$  can be explicitly constructed as follows:

$$\hat{h}(\theta, \phi) = (\sin \alpha(\theta, \phi) \cos \beta(\theta, \phi), \sin \alpha(\theta, \phi) \sin \beta(\theta, \phi), \cos \alpha(\theta, \phi)) \quad (44)$$

with

$$\begin{aligned}\alpha(\theta, \phi) &= 2 \cot^{-1}[\cot^n(\theta/2)], \\ \beta(\theta, \phi) &= n\phi.\end{aligned}\tag{45}$$

In the above equation,  $n$  is an integer that counts the number of times the image covers the target space. With this  $\hat{h}$ , the Pontrjagin index is  $\mathcal{P} = n$ . Consequently by Eq. (30) and Eq. (36) the Chern number is given by

$$C = \frac{1}{16\pi} \int_{S^2} d^2x \varepsilon^{\mu\nu} \hat{h} \cdot (\partial_\nu \hat{h} \times \partial_\mu \hat{h}) = \frac{\mathcal{P}}{2} = \frac{n}{2}.\tag{46}$$

For  $n > 1$  this gives the desired nontrivial bundle. Furthermore because the Pontrjagin index is a topological invariant, all maps having the same homotopy as that in Eq. (44) and Eq. (45) lead to the same Chern number [7].

Another way to modify Chern number is to keep  $\hat{h}$  the identity map but change the dimension of the  $\sigma$ 's. For example, if we replace the  $2 \times 2$  Pauli matrices by the  $3 \times 3$  matrices associated with the spin 1 representation of  $SU(2)$ :

$$\Sigma_x \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Sigma_y \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \Sigma_z \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},\tag{47}$$

then the Chern number  $C$  associated with the  $U(1)$  bundle whose fiber is the  $\lambda = +1$  eigenvectors of the resulting  $3 \times 3$  Hamiltonian is 1.

## B. Chern-Simons Invariant

The Chern number [Eq. (30)] records a specific topological property of fiber bundles. In the case of  $U(1)$  bundle it counts the number of monopoles (in the connection) enclosed by the (closed) 2D base space. If we have a  $U(1)$  bundle over a three dimensional base space whose connection has no monopole, the Chern number will vanish for all closed 2D surfaces. Under that condition the dual of the  $F_{\mu\nu}$ , i.e.,  $B^\mu = \epsilon^{\mu\nu\lambda} F_{\nu\lambda}$  must form closed loops in 3D. Now we encounter the next level of topological intricacy, namely, different  $B^\mu$  loops can *link* with one another!

This new level of topological non-triviality is reflected by the *Chern-Simons invariant* [13]:

$$CS = \int_{\mathcal{M}} Q_3(\mathcal{A}, \mathcal{F}) = \frac{1}{8\pi} \int_{\mathcal{M}} d^3x \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda}. \quad (48)$$

For non-Abelian connection (gauge field) the Chern-Simons invariant reads as

$$CS = \frac{1}{8\pi} \int_{\mathcal{M}} d^3x \epsilon^{\mu\nu\lambda} \text{Tr}[A_\mu F_{\nu\lambda}]. \quad (49)$$

It turns out that the Chern-Simons invariant is not quantized for the  $U(1)$  connection but is quantized ( $= m^2/8\pi$ ) for non-Abelian connections ( $m$  is an integer). It is important to note that  $CS$  is gauge-invariant for  $U(1)$  group if  $\mathcal{M}$  has no boundary. This follows from the identity  $\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0$  and the antisymmetry of  $\epsilon^{\mu\nu\lambda}$ .

In [7], Hsiang and Lee asked the interesting question: “Is it possible to construct simple  $2 \times 2$  Hamiltonian over 3-dimensional base space, so that the Chern-Simons invariant is non-zero?” In the following we reproduce their results.

**Example 5** Consider the three-sphere  $S^3$  parameterized by

$$\begin{aligned} S^3 = \{ & (\cos(t/2) \cos \alpha, \cos(t/2) \sin \alpha, \sin(t/2) \cos \beta, \sin(t/2) \sin \beta) \in \mathbb{R}^4; \\ & 0 \leq t \leq \pi, 0 \leq \alpha, \beta < 2\pi \}. \end{aligned} \quad (50)$$

Now consider the Hamiltonian

$$H = \hat{h}(t, \alpha, \beta) \cdot \vec{\sigma} \quad (51)$$

with  $\hat{h} : S^3 \rightarrow S^2$  being the nontrivial Hopf map specified by the following unit-vector function:

$$\hat{h}(t, \alpha, \beta) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (52)$$

where

$$\begin{aligned} \theta(t, \alpha, \beta) &= 2 \cot^{-1}[\cot^m(t/2)], \\ \phi(t, \alpha, \beta) &= m(\alpha - \beta). \end{aligned} \quad (53)$$

Here  $m$  is an integer. With this particular Hopf map, the Chern-Simons invariant in Eq. (48) computed from the Berry connection (of the +1 eigenstate) is  $CS = m^2/8\pi$  [7]. We will concentrate on this case hereafter.

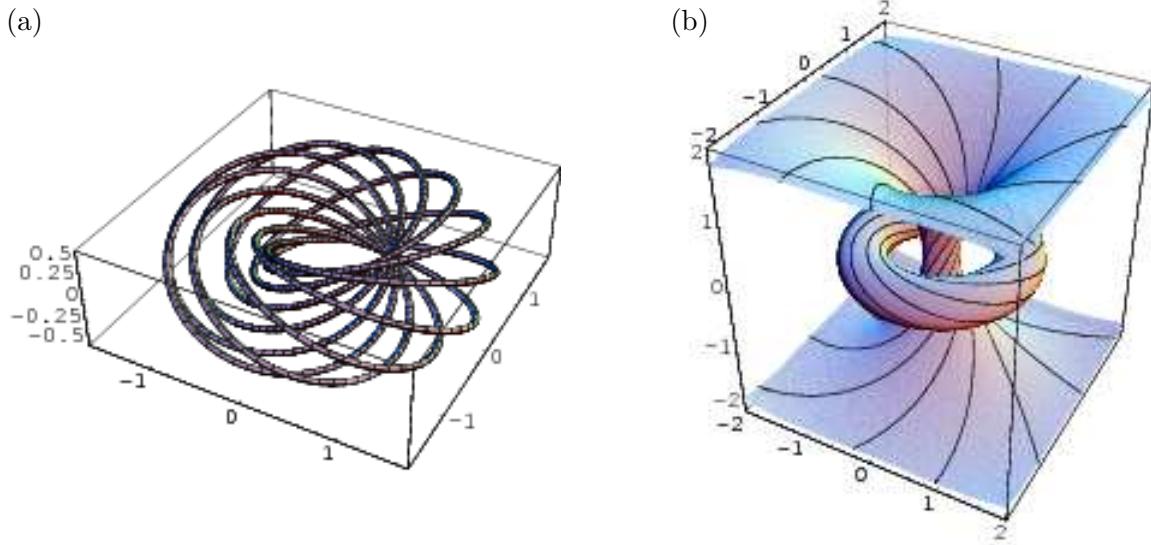


FIG. 1: The cluster composed of magnetic flux loops for  $\alpha - \beta = 0, 0.2\pi, 0.4\pi, \dots, 1.8\pi$  is shown in (a) with  $t = 0.3\pi$ . The magnetic loops are drawn with finite thickness in order to illustrate the mutual linking in perspective. The loops with the same  $t$  sweep out the surface of a torus. The tori with smaller  $t$ 's are inclosed by the tori with bigger  $t$ 's. Thus, every pair of loops is mutually linked. In particular, we draw the tori for  $t = 0.2\pi$  and  $t = 0.7\pi$  in (b).

*The dual of the Berry curvature (associated with the  $\lambda = +1$ ) is given by:*

$$B^\mu \equiv \varepsilon^{\mu\nu\lambda} F_{\nu\lambda} = \frac{1}{2} \varepsilon^{\mu\nu\lambda} \hat{h} \cdot (\partial_\lambda \hat{h} \times \partial_\nu \hat{h}). \quad (54)$$

Due to Eq. (53),  $\partial_\alpha \hat{h} = -\partial_\beta \hat{h}$ . As a result  $B^t = 0$  and  $B^\alpha = B^\beta$ . Consequently, the dual Berry curvature  $B^\mu$  is tangent to the one dimensional lines characterized by  $t = \text{constant}$  and  $\alpha - \beta = \text{constant}$ .

To visualize the flux lines associated with the connection, we employ the stereographic projection w.r.t the north pole ( $t = \pi, \beta = \pi/2$ ) of  $S^3$  and map the base space to  $R^3 \cup \{\infty\}$  by:

$$x = \frac{\cos(t/2) \cos \alpha}{1 - \sin(t/2) \sin \beta}, \quad y = \frac{\cos(t/2) \sin \alpha}{1 - \sin(t/2) \sin \beta}, \quad z = \frac{\sin(t/2) \cos \beta}{1 - \sin(t/2) \sin \beta}. \quad (55)$$

Since the stereographic projection is a conformal map, the dual Berry curvature  $B^\mu$  is still tangent to the lines specified by constant  $t$  and  $\alpha - \beta$  in  $(x, y, z)$ -coordinates. In Fig. 1, we show the loops formed by  $B^\mu$  lines with different values of  $t$  and  $\alpha - \beta$ . The manifold of  $B^\mu$  loops with the same  $t$  form the surface of a torus. All  $B^\mu$  loops on that surface mutually link with one another! In addition, a torus characterized by  $t = t_1$  is inclosed by another

torus  $t = t_2 > t_1$  such that their  $B^\mu$  loops also mutually link with one another. In sort the loops of the Berry curvature for the Hamiltonian specified by Eq. (51), Eq. (52), Eq. (53) and Eq. (55) have the fascinating property that every loop link with all other loops!

#### IV. THE SCHRÖDINGER EQUATION

Equipped with the above preliminaries, we are ready to study the Schrödinger equation on  $S^3$  in the presence of a magnetic field given by the dual Berry curvature  $B^\mu$  of Eq. (51), Eq. (52) and Eq. (53). In other words, the vector potential  $A_\mu$  that enters the Schrödinger equation is the Berry connection associated with the +1 vectors of Eq. (51), Eq. (52) and Eq. (53).

##### A. The Metric and Laplace Operator

Again, consider  $S^3$  as being embedded in  $R^4$  and parameterized by Eq. (50). The metric and Laplace operator on  $S^3$  are determined from those in  $R^4$ . With the coordinates in  $R^4$  given by Eq. (50) and the metric  $ds^2 = \sum_{\alpha=1}^4 dy_\alpha^2$ , we deduce the metric on  $S^3$  as

$$ds^2 = \frac{1}{4}dt^2 + \cos^2(t/2)d\alpha^2 + \sin^2(t/2)d\beta^2 \equiv g_{\mu\nu}dx^\mu dx^\nu; \quad (56)$$

i.e.

$$g_{\mu\nu} = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & \cos^2(t/2) & 0 \\ 0 & 0 & \sin^2(t/2) \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1/\cos^2(t/2) & 0 \\ 0 & 0 & 1/\sin^2(t/2) \end{pmatrix}. \quad (57)$$

The Laplace operator on  $S^3$  is deduced from Laplace operator in  $R^4$ , i.e.  $\sum_{\alpha=1}^4 \partial^2/\partial y_\alpha^2$ . The result is

$$g^{\mu\nu}[\partial_\mu\partial_\nu + (\partial_\mu \ln v)\partial_\nu] = 4\partial_t^2 + 4\left(\frac{\cos t}{\sin t}\right)\partial_t + \frac{1}{\cos^2(t/2)}\partial_\alpha^2 + \frac{1}{\sin^2(t/2)}\partial_\beta^2, \quad (58)$$

where  $v = (\sin t)/4$  is the Jacobian of the coordinate transformation in Eq. (50).

The Schrödinger equation in the presence of the magnetic field can be set up by replacing  $\partial_\mu \rightarrow \partial_\mu - iA_\mu^b$  in the Laplace operator. Thus we obtain

$$-\frac{1}{2M}g^{\mu\nu}[(\partial_\mu - iA_\mu^b)(\partial_\nu - iA_\nu^b) + (\partial_\mu \ln v)(\partial_\nu - iA_\nu^b)]\Psi = E\Psi, \quad (59)$$

where  $A_\mu^b$  is the Berry connection associated with the  $+1$  eigenstate  $|Z(t, \alpha, \beta)\rangle$  of Eq. (51), Eq. (52) and Eq. (53) via

$$A_\mu^b = -i\langle Z(t, \alpha, \beta) | \partial_\mu Z(t, \alpha, \beta) \rangle, \quad (60)$$

## B. Computing Berry Connection

To compute  $A_\mu^b$ , we choose the following gauge

$$|Z(t, \alpha, \beta)\rangle = \begin{pmatrix} e^{im\alpha} \cos[\theta(t)/2] \\ e^{im\beta} \sin[\theta(t)/2] \end{pmatrix}, \quad (61)$$

where  $\theta(t)$  is given by Eq. (53). It is easy to prove that the above  $|Z(t, \alpha, \beta)\rangle$  satisfies

$$H(t, \alpha, \beta) |Z(t, \alpha, \beta)\rangle = (+1) |Z(t, \alpha, \beta)\rangle \quad (62)$$

with  $H(t, \alpha, \beta)$  given by Eq. (51).

A simple calculation using Eq. (60) and Eq. (51) yields

$$A_t^b = 0, \quad A_\alpha^b = m \cos^2(\theta/2), \quad A_\beta^b = m \sin^2(\theta/2), \quad (63)$$

where  $\theta(t)$  is given by Eq. (53).

Substituting Eq. (57) and Eq. (63) into Eq. (59), we obtain the Schrödinger equation

$$\begin{aligned} -\frac{2}{M} \left\{ \partial_t^2 + \left( \frac{\cos t}{\sin t} \right) \partial_t + \frac{1}{4 \cos^2(t/2)} [\partial_\alpha - i m \cos^2(\theta/2)]^2 \right. \\ \left. + \frac{1}{4 \sin^2(t/2)} [\partial_\beta - i m \sin^2(\theta/2)]^2 \right\} \Psi = E \Psi. \end{aligned} \quad (64)$$

## V. SOLUTIONS WITHOUT MAGNETIC FIELD — $SO(4)$ SPHERICAL HARMONICS

In the absence of magnetic field, the Hamiltonian of Eq. (64) (with  $m = 0$ ) is invariant under  $SO(4)$  transformation. Consider the six generators of  $SO(4)$ :

$$\hat{L}_{ij} = r_i p_j - r_j p_i, \quad (65)$$

where  $\vec{r} = (y_1, y_2, y_3, y_4)$  and  $\vec{p} = -i(\partial_{y_1}, \partial_{y_2}, \partial_{y_3}, \partial_{y_4})$ . Each of  $\hat{L}_{ij}$  corresponds to the rotation on the plane spanned by  $y_i$  and  $y_j$ . Regrouping these operators as

$$\begin{aligned}\vec{L} &= (L_{23}, L_{31}, L_{12}), \\ \vec{M} &= (L_{14}, L_{24}, L_{34}),\end{aligned}\tag{66}$$

we obtain the following commutation relations:

$$\begin{aligned}[L_i, L_j] &= i\epsilon_{ijk}L_k \\ [M_i, L_j] &= i\epsilon_{ijk}M_k \\ [M_i, M_j] &= i\epsilon_{ijk}L_k.\end{aligned}\tag{67}$$

In the  $(t, \alpha, \beta)$ -coordinates, these operators read as

$$\begin{aligned}L_1 &= L_{23} = \frac{1}{i} \left\{ 2 \sin \alpha \cos \beta \partial_t - \tan \frac{t}{2} \cos \alpha \cos \beta \partial_\alpha - \cot \frac{t}{2} \sin \alpha \sin \beta \partial_\beta \right\}, \\ L_2 &= L_{31} = \frac{1}{i} \left\{ -2 \cos \alpha \cos \beta \partial_t - \tan \frac{t}{2} \sin \alpha \cos \beta \partial_\alpha + \cot \frac{t}{2} \cos \alpha \sin \beta \partial_\beta \right\}, \\ L_3 &= L_{12} = \frac{1}{i} \partial_\alpha,\end{aligned}\tag{68}$$

and

$$\begin{aligned}M_1 &= L_{14} = \frac{1}{i} \left\{ 2 \cos \alpha \sin \beta \partial_t + \tan \frac{t}{2} \sin \alpha \sin \beta \partial_\alpha + \cot \frac{t}{2} \cos \alpha \cos \beta \partial_\beta \right\}, \\ M_2 &= L_{24} = \frac{1}{i} \left\{ 2 \sin \alpha \sin \beta \partial_t - \tan \frac{t}{2} \cos \alpha \sin \beta \partial_\alpha + \cot \frac{t}{2} \sin \alpha \cos \beta \partial_\beta \right\}, \\ M_3 &= L_{34} = \frac{1}{i} \partial_\beta.\end{aligned}\tag{69}$$

A straightforward calculation using the above results gives

$$\begin{aligned}\vec{L}^2 + \vec{M}^2 &= L_1^2 + L_2^2 + L_3^2 + M_1^2 + M_2^2 + M_3^2 \\ &= - \left\{ 4\partial_t^2 + 4 \left( \frac{\cos t}{\sin t} \right) \partial_t + \frac{1}{\cos^2(t/2)} \partial_\alpha^2 + \frac{1}{\sin^2(t/2)} \partial_\beta^2 \right\},\end{aligned}\tag{70}$$

and

$$\vec{L} \cdot \vec{M} = \vec{M} \cdot \vec{L} = 0.\tag{71}$$

Eq. (70) implies that

$$-\nabla_{spherical}^2 = \vec{L}^2 + \vec{M}^2.\tag{72}$$

By introducing  $\vec{I} = (\vec{L} + \vec{M})/2$  and  $\vec{K} = (\vec{L} - \vec{M})/2$ , Eq. (67) becomes

$$\begin{aligned} [I_i, K_j] &= 0 \\ [I_i, I_j] &= i \epsilon_{ijk} I_k \\ [K_i, K_j] &= i \epsilon_{ijk} K_k. \end{aligned} \tag{73}$$

Consequently the Lie algebra  $so(4)$  is equivalent to  $so(2) \oplus so(2)$ , of which the representation states are  $|jm_I\rangle \otimes |jm_K\rangle$ . Furthermore due to Eq. (71),  $\vec{I}^2 = \vec{K}^2$ , thus  $j_I = j_K \equiv j$ . Therefore the representation states for  $SO(4)$  are

$$\begin{aligned} |jm_I\rangle \otimes |jm_K\rangle &\equiv |jm_I m_K\rangle, \\ m_i, m_j = -j, -j+1, \dots, j-1, j \\ j = 0, 1/2, 1, 3/2, \dots \end{aligned} \tag{74}$$

These states satisfy the following equations

$$\begin{aligned} \vec{I}^2 |jm_I m_K\rangle &= \vec{K}^2 |jm_I m_K\rangle = j(j+1) |jm_I m_K\rangle, \\ I_3 |jm_I m_K\rangle &= m_I |jm_I m_K\rangle, \\ K_3 |jm_I m_K\rangle &= m_K |jm_I m_K\rangle. \end{aligned} \tag{75}$$

Since  $-\nabla_{spherical}^2 = \vec{L}^2 + \vec{M}^2 = 2(\vec{I}^2 + \vec{K}^2)$ , the states  $|jm_I m_K\rangle$  are also the eigenstates of the Hamiltonian because

$$H = -\frac{1}{2M} \nabla_{spherical}^2 = \frac{1}{M} (\vec{I}^2 + \vec{K}^2), \tag{76}$$

hence

$$H |jm_I m_K\rangle = \frac{2}{M} j(j+1) |jm_I m_K\rangle. \tag{77}$$

The wavefunctions  $Y_j^{m_1, m_2}(t, \alpha, \beta)$  (defined below) corresponding to  $|jm_I m_K\rangle$  are the the 4-dimensional spherical harmonics. To drive them we first express the generators in  $(t, \alpha, \beta)$ -coordinates.

Firstly,  $I_3 = (L_3 + M_3)/2 = (\partial_\alpha + \partial_\beta)/2i$  and  $K_3 = (L_3 - M_3)/2 = (\partial_\alpha - \partial_\beta)/2i$  lead to  $-i\partial_\alpha = I_3 + K_3$  and  $-i\partial_\beta = I_3 - K_3$ ; hence

$$\begin{aligned} \frac{1}{i} \partial_\alpha |jm_I m_K\rangle &= (m_I + m_K) |jm_I m_K\rangle, \\ \frac{1}{i} \partial_\beta |jm_I m_K\rangle &= (m_I - m_K) |jm_I m_K\rangle. \end{aligned} \tag{78}$$

The solution of the above equations is given by the following

$$|jm_I m_K\rangle \equiv Y_j^{m_1, m_2}(t, \alpha, \beta) = P_j^{m_1, m_2}(t) e^{im_1\alpha} e^{im_2\beta}, \quad (79)$$

where

$$m_1 \equiv m_I + m_K, \quad m_2 \equiv m_I - m_K. \quad (80)$$

Next we define the raising and lowering operators,

$$\begin{aligned} L_{\pm} &= L_1 + iL_2 = \frac{e^{\pm i\alpha}}{i} \left\{ \mp 2i \cos \beta \partial_t - \tan \frac{t}{2} \cos \beta \partial_{\alpha} \pm i \cot \frac{t}{2} \sin \beta \partial_{\beta} \right\}, \\ M_{\pm} &= M_1 + iM_2 = \frac{e^{\pm i\alpha}}{i} \left\{ 2 \sin \beta \partial_t \mp i \tan \frac{t}{2} \sin \beta \partial_{\alpha} + \cot \frac{t}{2} \cos \beta \partial_{\beta} \right\}, \end{aligned} \quad (81)$$

and correspondingly,

$$\begin{aligned} I_{\pm} &= \frac{1}{2}(L_{\pm} + M_{\pm}) = \frac{e^{\pm i(\alpha+\beta)}}{2i} \left\{ \mp 2i \partial_t - \tan \frac{t}{2} \partial_{\alpha} + \cot \frac{t}{2} \partial_{\beta} \right\}, \\ K_{\pm} &= \frac{1}{2}(L_{\pm} - M_{\pm}) = \frac{e^{\pm i(\alpha-\beta)}}{2i} \left\{ \mp 2i \partial_t - \tan \frac{t}{2} \partial_{\alpha} - \cot \frac{t}{2} \partial_{\beta} \right\}. \end{aligned} \quad (82)$$

These operators satisfy

$$\begin{aligned} I_{\pm}|jm_I m_K\rangle &= [(j \mp m_I)(j \pm m_I + 1)]^{1/2} |j, m_I \pm 1, m_K\rangle \\ K_{\pm}|jm_I m_K\rangle &= [(j \mp m_K)(j \pm m_K + 1)]^{1/2} |j, m_I, m_K \pm 1\rangle. \end{aligned} \quad (83)$$

Now let us consider the highest-weight state with  $m_I = j$  and  $m_K = j$

$$|jjj\rangle = Y_j^{2j,0}(t, \alpha, \beta) = P_j^{2j,0}(t) e^{i2j\alpha} \quad (84)$$

Since this state is annihilated by  $I_+$  and  $K_+$  we require

$$\left( -2i \partial_t - \tan \frac{t}{2} \partial_{\alpha} \pm \cot \frac{t}{2} \partial_{\beta} \right) [P_j^{2j,0}(t) e^{i2j\alpha}] = 0, \quad (85)$$

or

$$\frac{d P_j^{2j,0}(t)}{dt} + j \tan \frac{t}{2} P_j^{2j,0}(t) = 0. \quad (86)$$

The solution of the above equation is

$$P_j^{2j,0}(t) \propto (\cos \frac{t}{2})^{2j}. \quad (87)$$

Hence the (normalized) highest weight state is given by

$$|jjj\rangle = Y_j^{2j,0}(t, \alpha, \beta) = \frac{1}{\pi} \sqrt{\frac{1+2j}{2}} \left( \cos \frac{t}{2} \right)^{2j} e^{2ij\alpha}. \quad (88)$$

Applying  $(I_-)^{j-m_I}(K_-)^{j-m_K}$  on  $Y_j^{2j,0}(t, \alpha, \beta)$  by Eq. (83), we generate  $Y_j^{m_1, m_2}(t, \alpha, \beta)$ . (Here we note  $[I_-, K_-] = 0$  hence the order of the operations does not matter.) In the following we list the first few  $Y_j^{m_1, m_2}(t, \alpha, \beta)$ :

$$j = 0 : \quad |0, 0, 0\rangle = Y_0^{0,0} = \frac{1}{\sqrt{2\pi}}; \quad (89)$$

$$\begin{aligned} j = 1/2 : \quad |1/2, 1/2, 1/2\rangle &= Y_{1/2}^{1,0} = \frac{1}{\pi} \cos \frac{t}{2} e^{i\alpha} \\ |1/2, 1/2, -1/2\rangle &= Y_{1/2}^{0,1} = -\frac{1}{\pi} \sin \frac{t}{2} e^{i\beta} \\ |1/2, -1/2, 1/2\rangle &= Y_{1/2}^{0,-1} = -\frac{1}{\pi} \sin \frac{t}{2} e^{-i\beta} \\ |1/2, -1/2, -1/2\rangle &= Y_{1/2}^{-1,0} = -\frac{1}{\pi} \cos \frac{t}{2} e^{-i\alpha}; \end{aligned} \quad (90)$$

$$\begin{aligned} j = 1 : \quad |1, 1, 1\rangle &= Y_1^{2,0} = \frac{1}{\pi} \sqrt{\frac{3}{2}} \cos^2\left(\frac{t}{2}\right) e^{2i\alpha} \\ |1, 1, 0\rangle &= Y_1^{1,1} = -\frac{\sqrt{3}}{2\pi} \sin t e^{i(\alpha+\beta)} \\ |1, 1, -1\rangle &= Y_1^{0,2} = \frac{1}{\pi} \sqrt{\frac{3}{2}} \sin^2\left(\frac{t}{2}\right) e^{2i\beta} \\ |1, 0, 1\rangle &= Y_1^{1,-1} = -\frac{\sqrt{3}}{2\pi} \sin t e^{i(\alpha-\beta)} \\ |1, 0, 0\rangle &= Y_1^{0,0} = -\frac{1}{\pi} \sqrt{\frac{3}{2}} \cos t \\ |1, 0, -1\rangle &= Y_1^{-1,1} = \frac{\sqrt{3}}{2\pi} \sin t e^{-i(\alpha-\beta)} \\ |1, -1, 1\rangle &= Y_1^{0,-2} = \frac{1}{\pi} \sqrt{\frac{3}{2}} \sin^2\left(\frac{t}{2}\right) e^{-2i\beta} \\ |1, -1, 0\rangle &= Y_1^{-1,-1} = \frac{\sqrt{3}}{2\pi} \sin t e^{-i(\alpha+\beta)} \\ |1, -1, -1\rangle &= Y_1^{-2,0} = \frac{1}{\pi} \sqrt{\frac{3}{2}} \cos^2\left(\frac{t}{2}\right) e^{-2i\alpha}. \end{aligned} \quad (91)$$

The wavefunction  $Y_j^{m_1, m_2}(t, \alpha, \beta)$  is an eigenstate of  $H$  with the eigenvalue  $E = 2j(j+1)/M$ . [See Eq. (77).]

## VI. SOLUTIONS WITH MAGNETIC FIELD OF NONZERO CHERN-SIMONS INVARIANT

### A. $m = 1$

Now, we return the Eq. (64) with  $m = 1$ . In this case, the Hopf map in Eq. (53) reads as

$$\begin{aligned}\theta(t, \alpha, \beta) &= t, \\ \phi(t, \alpha, \beta) &= \alpha - \beta,\end{aligned}\tag{92}$$

and the Berry connection in Eq. (63) simplifies to

$$A_t^b = 0, \quad A_\alpha^b = \cos^2(t/2), \quad A_\beta^b = \sin^2(t/2),\tag{93}$$

Substitute the above result into Eq. (64) we obtain

$$\begin{aligned}-\frac{2}{M} \left\{ \partial_t^2 + \left( \frac{\cos t}{\sin t} \right) \partial_t + \frac{1}{4 \cos^2(t/2)} [\partial_\alpha - i \cos^2(t/2)]^2 \right. \\ \left. + \frac{1}{4 \sin^2(t/2)} [\partial_\beta - i \sin^2(t/2)]^2 \right\} \Psi = E\Psi.\end{aligned}\tag{94}$$

In the presence of the vector potential the  $SO(4)$  symmetry is broken. However, Eq. (94) is still invariant under  $\alpha \rightarrow \alpha + a$ ,  $\beta \rightarrow \beta + b$ . As a result, we seek for the solution in the form of

$$\Psi(t, \alpha, \beta) = e^{i(m_1\alpha + m_2\beta)} \Theta(t).\tag{95}$$

Substitute the above expression into Eq. (94), we obtain

$$\lambda \Theta = \left\{ -\partial_t^2 - \left( \frac{\cos t}{\sin t} \right) \partial_t + \frac{m_1^2}{4 \cos^2(t/2)} + \frac{m_2^2}{4 \sin^2(t/2)} - \frac{1}{2}(m_1 + m_2) + \frac{1}{4} \right\} \Theta,\tag{96}$$

where  $\lambda = ME/2$ .

Let us define  $\lambda' = \lambda + (m_1 + m_2)/2 - 1/4$ , and reduce Eq. (96) to

$$\left\{ -\partial_t^2 - \left( \frac{\cos t}{\sin t} \right) \partial_t + \frac{m_1^2}{4 \cos^2(t/2)} + \frac{m_2^2}{4 \sin^2(t/2)} \right\} \Theta = \lambda' \Theta,\tag{97}$$

By substituting Eq. (79) into Eq. (64), it is simple to show that this is exactly the same differential equation satisfied by  $P_j^{m_1, m_2}(t)$  in the absence of the magnetic field ( $m = 0$ ), except that  $\lambda = ME/2$  is now replaced by  $\lambda'$ . Consequently,  $\Theta(t) = P_j^{m_1, m_2}(t)$  is till the

solution of Eq. (97), and  $Y_j^{m_1, m_2} = P_j^{m_1, m_2}(t) e^{im_1\alpha} e^{im_2\beta}$  is still the function of  $H$ . In other words

$$H_{m=1} Y_j^{m_1, m_2}(t, \alpha, \beta) = E Y_j^{m_1, m_2}(t, \alpha, \beta), \quad (98)$$

where

$$E = \frac{2}{M} \left[ j(j+1) - \frac{m_1 + m_2}{2} + \frac{1}{4} \right]. \quad (99)$$

### B. General $m$

Like the  $m = 1$  case, Eq. (64) is still invariant under  $\alpha \rightarrow \alpha + a$ ,  $\beta \rightarrow \beta + b$ , hence the solution has the form of Eq. (95). Unfortunately, for  $m \neq 0, 1$  the equation of  $\Theta(t)$  is quite formidable.

However, as discussed by Hsiang and Lee [7], so long as the map  $\hat{h}(t, \alpha, \beta)$  used in Eq. (51) is in the same homotopy class as the map in Eq. (53), the Chern-Simons invariant computed from the Berry connection via Eq. (48) remain unchanged. This allow us to deform the map in Eq. (53) to a simpler map while preserving its homotopy class:

$$\begin{aligned} \theta(t, \alpha, \beta) &= t, \\ \phi(t, \alpha, \beta) &= m(\alpha - \beta). \end{aligned} \quad (100)$$

This deformed Hopf map is continuous and respects the bijection (one-to-one and onto) from  $t$  to  $\theta$ . As a result, it is in the same homotopy class as the map in Eq. (53).

With this map, the magnetic potential computed from the Berry connection is given by

$$A_t^b = 0, \quad A_\alpha^b = m \cos^2(t/2), \quad A_\beta^b = m \sin^2(t/2). \quad (101)$$

Compare this result with Eq. (93) we notice that the new vector potential is  $m$  times that of the  $m = 1$  case. (Since the Chern-Simons invariant is quadratic in  $A_\mu^b$  this increases  $CS$  by a factor of  $m^2$ .) When such vector potential is substituted into Eq. (64) everything is the same as  $m = 1$  case except an overall factor  $m$  is involved. Therefore, the differential equation for  $\Theta(t)$  reads

$$\left\{ -\partial_t^2 - \left( \frac{\cos t}{\sin t} \right) \partial_t + \frac{m_1^2}{4 \cos^2(t/2)} + \frac{m_2^2}{4 \sin^2(t/2)} + \frac{m(m - 2m_1 - 2m_2)}{4} \right\} \Theta = \lambda \Theta. \quad (102)$$

Compare the above equation with Eq. (97) we note that they only differ by a redefinition of  $\lambda'$ . As the result the eigenstates and eigenvalues of  $H(m)$  are

$$H(m)|j m_I m_K\rangle = E|j m_I m_K\rangle, \quad E = 2\lambda/M, \\ \lambda = j(j+1) - \frac{m}{2}(m_1 + m_2) + \frac{m^2}{4}. \quad (103)$$

From Eq. (103), we see that the effect of the self-linking magnetic field is to lift the energy level degeneracy while preserving the eigen-wavefunctions. This is very similar to the Zeeman effect in elementary quantum mechanics.

We can easily see why this is so by expressing  $H$  in terms of  $SO(4)$  operators:

$$H \equiv -\frac{2}{M} \left\{ \partial_t^2 + \left( \frac{\cos t}{\sin t} \right) \partial_t + \frac{1}{4 \cos^2(t/2)} [\partial_\alpha - i m \cos^2(\theta/2)]^2 \right. \\ \left. + \frac{1}{4 \sin^2(t/2)} [\partial_\beta - i m \sin^2(\theta/2)]^2 \right\} \\ = -\frac{2}{M} \left\{ \partial_t^2 + \left( \frac{\cos t}{\sin t} \right) \partial_t + \frac{\partial_\alpha^2}{4 \cos^2(t/2)} + \frac{\partial_\beta^2}{4 \sin^2(t/2)} - \frac{i}{2} m(\partial_\alpha + \partial_\beta) - \frac{m^2}{4} \right\} \\ = -\frac{2}{M} \left\{ \frac{\nabla_{spherical}^2}{4} - \frac{i}{2} m(\partial_\alpha + \partial_\beta) - \frac{m^2}{4} \right\} \\ = \frac{2}{M} \left\{ \frac{1}{2} (\vec{I}^2 + \vec{K}^2) - m I_3 + \frac{m^2}{4} \right\}. \quad (104)$$

The above equation makes it clear that  $|j m_I m_K\rangle$  are eigenstates.

## VII. DISCUSSION

In this paper, we have solved the Schrödinger equation for a charged particle moving on a *3-sphere* ( $S^3$ ) under the influence of a magnetic field whose flux lines exhibit mutual linking. (The Chern-Simons invariant associated with the vector potential is non-zero.)

The vector potential of the self-linking magnetic field is obtained from the Berry connection of a simple  $2 \times 2$  Hamiltonian  $H = \hat{h} \cdot \vec{\sigma}$ , where  $\hat{h}$  is a three-component unit vector field over  $S^3$ . It turns out that when  $\hat{h}$  corresponds to a topologically non-trivial mapping from  $S^3$  to  $S^2$ , the associated Berry's connection exhibits a non-zero Chern-Simons invariant. For simple choice of  $\hat{h}$ , e.g., the map given in Eq. (100), the eigenstates are  $SO(4)$  spherical harmonics  $Y_j^{m_1, m_2}$ . However the degeneracy of eigenenergy in zero field is lifted in a manner reminiscent of the Zeeman effect.

This result is very different from the case of a charged particle moving on a *two sphere* under the influence of a Dirac monopole. In the latter case the  $SO(3)$  spherical harmonics  $Y_{l,m}$  are not the eigenfunctions. The real eigen solutions, the monopole harmonics  $Y_{q,l,m}$ , have *section* structure. They are the basis for the irreducible representations of  $SU(2)$  which is the universal covering group of  $SO(3)$ . This originates from the fact that there does not exist an everywhere non-singular vector potential for the monopole field over the entire two-sphere. On the contrary, there does exist a non-singular vector potential describing a self-linking magnetic field over the entire  $S^3$ .

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